Linear algebra I v.s. II

I	II
Euclidean space	Vector space
$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n.$	V (abstract)
Matrix	Linear transformation
$M_{m\times n}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$	$T: V^n \longrightarrow W^m$
$\vec{x} \rightsquigarrow \vec{y} = M_{m\times n} \cdot \vec{x}$	

Finite dimension in both courses

§ Vector Spaces

Notation: $\mathbb{R}$ = set of real numbers

$\mathbb{C}$ = set of complex numbers

$+, -, \times, /$

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (Generally, $\mathbb{F}$ denotes a field)

Def: A vector space over $\mathbb{F}$ is a set equipped with two operations.

- addition $+ : V \times V \longrightarrow V$ $\quad (\vec{x}, \vec{y}) \to \vec{x} + \vec{y}$.
- Scalar multiplication $\cdot : \mathbb{F} \times V \longrightarrow V$ $\quad (a, \vec{x}) \to a\vec{x}$

Satisfying the following properties:

about $+$

(VS 1) $\vec{x}+\vec{y} = \vec{y}+\vec{x} \quad \forall \vec{x}, \vec{y} \in V$ (Comm.)

(VS 2) $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z}) \quad \forall \vec{x}, \vec{y}, \vec{z} \in V$. (ass.)

(VS 3). $\exists\, \vec{0} \in V$ s.t. $\vec{x}+\vec{0} = \vec{x} \quad \forall \vec{x} \in V$. (zero)

(VS 4) $\forall \vec{x} \in V, \ \exists\, \vec{y} \in V$ s.t. $\vec{x}+\vec{y} = \vec{0}$ (inverse)

about $\cdot$

(VS 5) $1 \cdot \vec{x} = \vec{x} \quad \forall \vec{x} \in V$. (unit)

(VS 6) $(ab)\vec{x} = a(b\vec{x}) \quad \forall a, b \in F, \ \vec{x} \in V$. (assoc.)

about

(VS 7) $a(\vec{x}+\vec{y}) = a\vec{x} + a\vec{y} \quad \forall a \in F, \ \vec{x}, \vec{y} \in V$. (dist.)

(VS 8) $(a+b)\vec{x} = a\vec{x} + b\vec{x} \quad \forall a, b \in F, \ \vec{x} \in V$. (dist.)

Examples:

- $\boxed{F^n = \{(x_1, \cdots, x_n) : x_j \in F \text{ for } j = 1, \cdots, n\}}$

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) := (x_1 + y_1, \cdots, x_n + y_n)$$

$$a \cdot (x_1, \cdots, x_n) := (a x_1, a x_2, \cdots, a x_n)$$

- $\boxed{M_{m \times n}(F) = \{m \times n \text{ matrices w/ entries in } F\}}$

w/ matrix addition and scalar multip.

- $\boxed{P(F) = \{\text{polynomials w/ coeff in } F\}}$. also denoted $F[x]$.

w/ poly addition and scalar multip.

- $\boxed{P_n(\mathbb{F}) = \{\text{poly w/ coeff in } \mathbb{F} \text{ with degree} \le n\}}$

- Let S be any nonempty set.

  Then $\boxed{\mathcal{F}(S, \mathbb{F}) = \{\text{function } f: S \to \mathbb{F}\}}$, also denoted $\mathbb{F}^S$ is a vector space over $\mathbb{F}$:

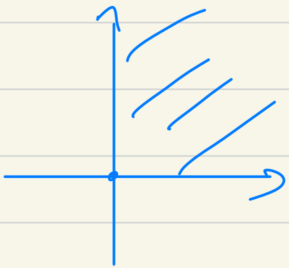
$$(f+g)(s) := f(s) + g(s) \quad ; \quad (af)(s) = a \cdot f(s) \quad \forall s \in S.$$

- $\boxed{\mathbb{F}^\infty = \{(x_1, x_2, \cdots): x_j \in \mathbb{F} \ \forall j = 1, 2, \cdots\} = \{\text{seq. of elt. in } \mathbb{F}\}}$

Non-Examples:

- $\mathbb{R}^2 - \{0\}$ $\quad \vec{0}$, $\mathbb{R}^2 - \{\vec{x}\}$

- First quadrant of $\mathbb{R}^2$. inverse



- $\{$poly w/ coeff in F, degree $= n > 0\}$ $\quad (x^n) + (-x^n + 1) = 1$

- $F = \mathbb{C}$, $V = \mathbb{R}^n$

  equipped with $c \cdot \vec{x} := |c| \cdot \vec{x}$ $\quad$ (VS 8): $(a+b)\cdot\vec{x} = a\vec{x} + b\vec{x}$

  $|a+b| \neq |a| + |b|$ in general!

Proposition: Let V be a vector space over F. Then

(a). The element $\vec{0}$ in (VS 3) is unique, called the zero vector in V.

(b) $\forall \vec{x} \in V$, the element $\vec{y}$ in (VS 4) is unique, called the additive inverse of $\vec{x}$. and denoted as $-\vec{x}$.

(c) $\vec{x} + \vec{z} = \vec{y} + \vec{z} \Rightarrow \vec{x} = \vec{y}$ (Cancellation law).

(d) $0 \cdot \vec{x} = \vec{0} \quad \forall \vec{x} \in V$.

(e) $a \cdot \vec{0} = \vec{0} \quad \forall a \in F$.

(f) $(-a)\vec{x} = -(a\vec{x}) = a(-\vec{x}) \quad \forall a \in F, \vec{x} \in V$.

Pf: (a). If $\vec{0}, \vec{0}' \in V$ are two elts satisfying (VS 3)

then $\vec{0} = \vec{0} + \vec{0}' = \vec{0}'$.

(b). Given $\vec{x} \in V$. Suppose we have $\vec{y}, \vec{y}' \in V$ satisfying (VS 4).

i.e., $\vec{x} + \vec{y} = \vec{0} = \vec{x} + \vec{y}'$

$$\Rightarrow \quad \vec{y} = \vec{y} + \vec{0} = \vec{y} + \cancelto{\vec{0}}{(\vec{x} + \vec{y}')} = \cancelto{\vec{0}}{(\vec{y} + \vec{x})} + \vec{y}' = \vec{0} + \vec{y}' = \vec{y}'$$

(c). $\vec{x} + \vec{z} = \vec{y} + \vec{z} \Rightarrow \vec{x} + \cancelto{\vec{0}}{(\vec{z} + (-\vec{z}))} = \vec{y} + \cancelto{\vec{0}}{(\vec{z} + (-\vec{z}))} \Rightarrow \vec{x} = \vec{y}$

(d). $0\vec{x} = (0+0)\vec{x} = 0\vec{x} + 0\vec{x} \Rightarrow 0\vec{x} = \vec{0}$.

(e). (f) are left as exercises. $\square$

§ Subspaces.

Def: W is a **subspace** of a vector space V if
(1) $W \subset V$.
(2). W vector space with the same $+$ and $\cdot$.

Instead of checking 8 axioms, only need the following.

Prop: Let V be a vector space. A subset $W \subset V$ is a subspace iff the following 3 conditions hold for the operations defined on V.

(a). $\vec{0} \in W$.
(b). $\vec{x} + \vec{y} \in W \quad \forall \vec{x}, \vec{y} \in W$. (closed under addition)
(c). $a\vec{x} \in W \quad \forall a \in F$ and $\vec{x} \in W$. (closed under scalar mult.)

Pf: ($\Rightarrow$) If $W \subset V$ is a subspace, then (b) and (c) must hold because the operations are well-defined on W.

Also, W has a zero vector $\vec{0}_W$.

But then $\vec{0}_W + \vec{0}_W = \vec{0}_W$ in V.

$\Rightarrow$ $\vec{0}_W = \vec{0}$. the zero vector in V.

($\Leftarrow$) If (a) - (c) hold, then $+$ and $\cdot$ are well-defined on W.

(VS 3) follows from (a)

(VS 1), (VS 2), (VS 5 – VS 8) hold for V, so they hold for W.

It remains to check (VS 4)

Let $\vec{x} \in W$. then $-\vec{x} \in V$. But $-\vec{x} = (-1)\vec{x} \in W$ by (c). $\square$

Examples.

trivial subspace

- For any vector space V, $\{\vec{0}\} \subset V$, and $V \subset V$ are subspaces.

- Consider $V = F^n$. Then any system of linear equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_n \end{cases}$$

where $a_{ij}, b_i \in F$

Solution set $S \subset V$ is a subspace of F^n iff $\vec{b} = 0$.

$(\Rightarrow)$: $0 \in S \Rightarrow \vec{b} = 0$.

$(\Leftarrow)$: $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n) \rightsquigarrow (x_1+y_1, \cdots, x_n+y_n)$ sol

- For $V = M_{n\times n}(F)$.

  $W_1 = \{\text{diagonal matrices}\} \subset V$.

  $W_2 = \{ A \in M_{n\times n}(F) ; \ tr(A) = 0 \}$ where $tr(A) := \sum_{i=1}^{n} a_{ii}$

  are subspaces.

- $P_n(F)$ subspace of $P(F)$

- For $V = \mathcal{F}(\mathbb{R}, \mathbb{R}) = \{\mathbb{R}\text{-valued functions on } \mathbb{R}\}$

  $C(\mathbb{R}) := \{\text{Continuous functions}\} \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$ is a subspace.

Non-Example:

What happens to union and intersection of Subspace?

Prop: Any intersection of subspaces of a vector space V is a subspace of V.

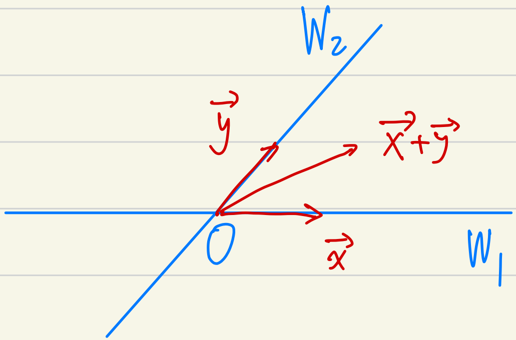
Pf: Let $\{W_i\}_{i\in I}$ be a collection of subspaces of V.

Set $W = \bigcap_{i\in I} W_i$

- Since $\vec{0} \in W_i$ $\forall i \in I$. we have $\vec{0} \in W$.
- For $\vec{x}, \vec{y} \in W$, we have $\vec{x}, \vec{y} \in W_i$ $\forall i \in I$

  $\Rightarrow \vec{x}+\vec{y} \in W_i$ $\forall i \in I$. $\Rightarrow \vec{x}+\vec{y} \in W$.
- For $\vec{x} \in W$ and $a \in F$ $a\vec{x} \in W_i$ since $\vec{x} \in W_i$ $\forall i \in I$. $\Rightarrow a\vec{x} \in W$. $\square$

Union of subspaces is generally not a subspace.



not closed under addition.

Def: · W_1, W_2 subspace of V. Sum of W_1 and W_2 is

$$W_1 + W_2 := \{\vec{x}+\vec{y} : \vec{x}\in W_1, \vec{y}\in W_2\}$$

is a vector subspace

· When $W_1 \cap W_2 = \{0\}$, the above is called direct sum, denoted $W_1 \oplus W_2$.

§ Linear Combinations and Span

Def: Let V be a vector space over F, $S \subset V$ is a nonempty subset.

- A vector $\vec{v} \in V$ is a linear combination of vectors of S ← $|S| = \infty$ possible
  if $\exists\ \vec{u}_1, \cdots, \vec{u}_n \in S$ and $a_1, \cdots, a_n \in F$. s.t.

$$\vec{v} = a_1 \vec{u}_1 \cdots + a_n \vec{u}_n .$$

← Coefficient

- The span of S, denoted span(S), is the set of all linear combs of vectors of S.

$$\text{span}(S) := \{ a_1\vec{u}_1 + \cdots + a_n\vec{u}_n \cdot a_j \in F,\ \vec{u}_j \in S \text{ for } j = 1, \cdots, n \}$$

Remark: · All linear combination contain <u>finitely many</u> terms !

· Convention: If $S=\phi$. $\operatorname{span}\phi := \{0\}$.

<u>Examples</u>. · $F^n = \operatorname{span}\{\vec{e_1}, \dots, \vec{e_n}\}$ where $\vec{e_j} = (0, \dots \underset{j\text{th}}{1}, \dots 0)$

· $1 \in \operatorname{Span}\{1+x^2, 1-x^2\}$ but $x \notin \operatorname{span}\{1+x^2, 1-x^2\}$

· $M_{2\times2}(F) = \operatorname{span}\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

· $P(F) = \operatorname{span}\{1, x, x^2, \dots\}$ $\quad \sum_{i=0}^{\infty} x^i = 1+x+x^2+\dots \notin P(F)$

Theorem: Let $S \subset V$ be a subset of a vector space V over F.
Then $\mathrm{Span}(S)$ is the smallest subspace of V containing S.

Pf: If $S = \phi$, then $\mathrm{span}(\phi) = \{0\}$.

Suppose $S \neq \phi$.

- Let $\vec{z} \in S$. Then $\vec{0} = 0 \cdot \vec{z} \in \mathrm{Span}(S)$.
- If $\vec{x}, \vec{y} \in \mathrm{Span}(S)$, then we can write
$$\vec{x} = a_1 \vec{u_1} + \cdots + a_m \vec{u_m} \quad \text{and} \quad \vec{y} = b_1 \vec{v_1} + \cdots + b_n \vec{v_n}$$
where $\vec{u_1}, \cdots, \vec{u_m}, \vec{v_1}, \cdots, \vec{v_n} \in S$, $a_1, \cdots, a_m, b_1, \cdots, b_n \in F$.
So $\vec{x} + \vec{y} = a_1 \vec{u_1} + \cdots + a_m \vec{u_m} + b_1 \vec{v_1} + \cdots b_n \vec{v_n} \in \mathrm{Span}(S)$.
- $\forall c \in F$. $c\vec{x} = (ca_1)\vec{u_1} + \cdots + (ca_m)\vec{u_m} \in \mathrm{Span}(S)$

Span(S) is a subspace.

Now let $W \subset V$ subspace containing S.

Want to show $\mathrm{span}(S) \subset W$.

Let $\vec{x} \in \mathrm{span}(S)$. We can write $\vec{x} = a_1\vec{u_1} + \cdots + a_m\vec{u_m}$, $\vec{u_1}, \cdots, \vec{u_m} \in S$.

As $S \subset W$, $\vec{u_1}, \cdots, \vec{u_m} \in W$.

Closed under addition & scalar mult

As W is a subspace, we have $\vec{x} = a_1\vec{u_1} + \cdots + a_m\vec{u_m} \in W$.

Hence $\mathrm{Span}(S) \subset W$. $\Rightarrow$ $\mathrm{Span}(S)$ the smallest subspace containing S.

$\square$

Def: A subset $S \subset V$ **spans** (or **generates**) V if $\mathrm{span}(S) = V$.
We also say S is a **spanning set** (or **generating set**) for V.

e.g.

- $\{\vec{e_1}, \cdots, \vec{e_n}\}$ spans F^n. $e_i = (0, \cdots \overset{i^{th}}{1}, 0, \cdots 0)$
- $\{1, x, x^2, \cdots\}$ spans $P(F)$
- $\mathcal{F}(\mathbb{R}, \mathbb{R})$ or $C(\mathbb{R})$ hard to describe an explicit spanning set

$\mathbb{R}$-valued f^n on $\mathbb{R}$ Cont. funct. $\{1, x, x^2, \cdots, \sin x, \cos x, e^x, \ln x, \cdots\}$